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Quaternionic solutions for the relativistic Kepler problem with magnetic charges†

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Abstract. The relativistic Kepler problem in Dirac form is solved by means of a quaternionic method for particles having both electric and magnetic charges. The variables in spherical coordinates can be separated in a simple way and the results obtained are similar to those which are valid for the usual hydrogen atom.

1. Introduction

Although the concept of charged magnetic monopoles was introduced many years ago (Dirac 1931) in quantum theory, the relativistic Kepler problem with magnetic charges has been considered in detail only recently. First, Berrondo and McIntosh (1970) investigated the symmetry and degeneracy of the Dirac equation for a Coulomb potential with a fixed centre bearing both electric and magnetic charges; the formalism used was closely related to the 'symmetric' Hamiltonian of Biedenharn (1962) and Biedenharn and Swamy (1964) and the accidental doubling degeneracy was deduced in a way using essentially the algebra of Malkin and Manko (1969). More recently, Barut and Bornzin (1970) gave a SO(4, 2) formulation of a similar problem and deduced the spectrum from group theoretical considerations only. These formalisms are very efficient in exhibiting the symmetry properties of the problem and they provide results applicable to the theory of strong interaction phenomena based on the concept of magnetic charges (Barut 1971), but they do not allow us to put the wavefunctions into a convenient form for practical calculations. Now, it was shown recently by Hautot (1970) that the use of quaternions allows a direct integration of the Dirac equation for the hydrogen atom, by separation of variables. The object of this paper is to show that the quaternionic method is well adapted to the study of the relativistic Kepler problem with magnetic charges, and that it allows us to deduce the corresponding wavefunctions from those of the well known ordinary relativistic Kepler problem in a simple way.

2. Separation of variables

If we consider the problem of a particle with mass m, electric charge e_1 and magnetic charge g_1 moving in the field of another particle with infinite mass, electric charge e_2 and magnetic charge g_2 situated at the origin, the corresponding Dirac equation can

† After this paper was written, we received a preprint of A Hautot (to be published in *J. math. Phys.*) in which a particular case of this problem was treated.

be written (Barut and Bornzin 1970)

$$\left(c(\boldsymbol{\alpha}\,.\,\boldsymbol{\pi})+\gamma_0 mc^2-\frac{\alpha}{r}\right)\psi = E\psi \tag{1}$$

where

$$\boldsymbol{\pi} = \boldsymbol{p}-\mu\boldsymbol{D}(\boldsymbol{r})$$

$$\mu = e_1 g_2 - g_1 e_2$$

$$\alpha = -(e_1 e_2 + g_1 g_2)$$

$$\boldsymbol{D}(\boldsymbol{r}) = \frac{\boldsymbol{r}\times\boldsymbol{n}(\boldsymbol{r}\,.\,\boldsymbol{n})}{r\{r^2-(\boldsymbol{r}\,.\,\boldsymbol{n})^2\}}$$

$\boldsymbol{n}$ being an arbitrary unit vector.

Using the notations of Hautot (1970), equation (1) in quaternionic form reads

$$\begin{aligned}\left(\nabla_3-\frac{\mu}{c\hbar}\sqrt{-1}\mathrm{D}_3\right)u &= -u\left\{\frac{mc}{\hbar}j+\sqrt{-1}i\left(\frac{E}{c\hbar}+\frac{\alpha}{c\hbar r}\right)\right\}\\ &= -uQ(r)\end{aligned} \tag{2}$$

where

$$\psi = \exp\{-(\sqrt{-1}/\hbar)Et\}u$$

$$\nabla_3 = i\partial_x + j\partial_y + k\partial_z$$

and

$$\mathrm{D}_3 = i\mathrm{D}_x + j\mathrm{D}_y + kD_z.$$

As in the case of the usual hydrogen atom ($\mu = 0$), it is convenient to use spherical coordinates which are separable; then

$$\nabla_3 = k\,\mathrm{e}^{k\phi/2}\,\mathrm{e}^{-j\theta}\,\mathrm{e}^{-k\phi/2}\partial_r + i\,\mathrm{e}^{-k\phi/2}\,\mathrm{e}^{-j\theta}\,\mathrm{e}^{-k\phi/2}\frac{1}{r}\partial_\theta + j\,\mathrm{e}^{-k\phi}\frac{1}{r\sin\theta}\partial_\phi \tag{3}$$

and if we choose $\boldsymbol{n}$ parallel to the z axis

$$\mathrm{D}_3 = -j\frac{\cot\theta}{r}\mathrm{e}^{-k\phi}. \tag{4}$$

Following the method described by Hautot (1970), we find that the separation of variables holds in the form

$$u \equiv \exp\{\tfrac{1}{2}(k+2\sqrt{-1}M)\phi\}\,\mathrm{e}^{j\theta/2}\Theta(\theta)R(r) \tag{5}$$

if $[\Theta, k] = 0$, which implies that $\Theta = \Theta_1 + k\Theta_4$; since $u(r,\theta,\phi) = u(r,\theta,\phi+2\pi)$, M must be a half-odd integer and the equations satisfied respectively by $\Theta(\theta)$ and $R(r)$ are

$$j\left(\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}+\frac{1}{2}\cot\theta\Theta\right)-i\frac{\sqrt{-1}}{\sin\theta}\left(\frac{\mu}{c\hbar}\cos\theta+M\right)\Theta = \Theta A \tag{6}$$

$$r\frac{\mathrm{d}R}{\mathrm{d}r}-rkRQ = (A-1)R \tag{7}$$

where the quaternion A, which plays the role of a separation constant, is equal to $ai+bj$, where a and b are scalar quantities; however we can put $b = 0$ without loss of generality.

3. Quaternionic solutions

3.1. Angular variables

Since $\Theta = \Theta_1 + k\Theta_4$, equation (6) is equivalent to a system of two scalar coupled differential equations, and this system can be expressed in real form by putting

$$Y_4 = -\Theta_1 + \sqrt{-1}\Theta_4$$
$$Y_1 = \pm(\Theta_1 + \sqrt{-1}\Theta_4)$$

and

$$\kappa = \mp a\sqrt{-1}.$$

More precisely, Y_1 and Y_4 satisfy the following equations:

$$\frac{d}{d\theta}Y_1 + \frac{1}{2}\cot\theta\, Y_1 + \frac{1}{\sin\theta}\left(\frac{\mu}{c\hbar}\cos\theta + M\right)Y_1 = \kappa Y_4$$
$$\frac{d}{d\theta}Y_4 + \frac{1}{2}\cot\theta\, Y_4 - \frac{1}{\sin\theta}\left(\frac{\mu}{c\hbar}\cos\theta + M\right)Y_4 = -\kappa Y_1. \tag{8}$$

By decoupling the system (8), we obtain the second order differential equations

$$\frac{d^2}{d\theta^2}Y_i + \cot\theta\frac{d}{d\theta}Y_i + \left(\sigma_i - \frac{(M - K_i\cos\theta)^2}{\sin^2\theta}\right)Y_i = 0 \qquad i = 1, 4 \tag{9}$$

where

$$\sigma_1 = -\frac{1}{2} - \frac{\mu}{c\hbar} + \kappa^2 \qquad K_1 = -\frac{1}{2} - \frac{\mu}{c\hbar}$$

and

$$\sigma_4 = -\frac{1}{2} + \frac{\mu}{c\hbar} + \kappa^2 \qquad K_4 = \frac{1}{2} - \frac{\mu}{c\hbar}.$$

Equation (9) is exactly the same as encountered in the symmetric top problem, but here, neither M nor K is an integer. The easiest way to solve it is to use the factorization method (A-type); in fact, the results obtained by Infeld and Hull (1951) for the symmetric top are still valid, and we can get quadratically integrable solutions only if $K \pm M$ is an integer. M being a half-odd integer, equations (9) show that $\mu/c\hbar$ must be an integer which is nothing else but the Dirac quantization condition (Dirac 1948 and Hurst 1968). In this case, the solutions can be expressed in terms of Jacobi polynomials and are labelled by a quantum number $\mathscr{J}$ defined by

$$\left(\mathscr{J} + \frac{1}{2}\right)^2 = \sigma + K^2 + \frac{1}{4} = \kappa^2 + \left(\frac{\mu}{c\hbar}\right)^2$$
$$\mathscr{J} - |M|, \mathscr{J} - |K| = 0, 1, 2, \ldots. \tag{10}$$

M and K being half-odd integers, $\mathscr{j}$ must be a positive half-odd integer. For a given value of $\mathscr{j}$, there are two distinct solutions for Θ according to the sign of κ. More precisely, changing the sign of κ is equivalent to multiplying Θ by $k\sqrt{-1}$.

3.2. *Radial variables*

Formally, equation (7) is the same as for the usual hydrogen atom ($\mu = 0$); the only difference is that in general $\kappa = -a\sqrt{-1}$ is no longer an integer. However this point is not essential in the resolution of radial equations obtained for the usual hydrogen atom (Bethe and Salpeter 1957), and all the results we need can be derived from that case by replacing $(\mathscr{j}+\frac{1}{2})^2$ by $(\mathscr{j}+\frac{1}{2})^2-(\mu/c\hbar)^2$ in the formulae. In particular it allows us to find the energy spectrum

$$E_n = mc^2\left\{1+\left(\frac{\alpha}{c\hbar}\right)^2\left[n+\left\{\left(\mathscr{j}+\frac{1}{2}\right)^2-\left(\frac{\alpha}{c\hbar}\right)^2-\left(\frac{\mu}{c\hbar}\right)^2\right\}^{1/2}\right]^{-2}\right\}^{-1/2} \tag{11}$$

for a small coupling constant

$$\left(\frac{\alpha}{c\hbar}\right)^2 < \left(\mathscr{j}+\frac{1}{2}\right)^2-\left(\frac{\mu}{c\hbar}\right)^2$$

if g_1g_2 is not zero, $(\alpha/c\hbar)^2$ is large ($>137/4$) then, we must start from the results given by Case (1950).

In any case, as for the usual hydrogen atom, the radial part of the wavefunction can be written (Hautot 1970)

$$R = (R_1+iR_2)(1-j) \tag{12}$$

where R_1 and R_2 are scalar quantities; the system of radial equations may be expressed in real form by putting

$$G = r(R_1+\sqrt{-1}R_2)$$
$$F = -r(R_2+\sqrt{-1}R_1)$$

we obtain the usual system of two coupled differential equations (Bethe and Salpeter 1957)

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}r}+\frac{\kappa}{r}G &= \frac{1}{c\hbar}\left((mc^2+E)+\frac{\alpha}{r}\right)F\\ \frac{\mathrm{d}F}{\mathrm{d}r}-\frac{\kappa}{r}F &= \frac{1}{c\hbar}\left((mc^2-E)-\frac{\alpha}{r}\right)G\end{aligned} \tag{13}$$

where F and G are respectively the small and the large component, but here κ is not necessarily an integer.

4. Invariants

It is well known (Berrondo and McIntosh 1970) that the three components of the angular momentum operator

$$\mathscr{J} = \mathscr{L}+\boldsymbol{S} \tag{14}$$

where $\mathscr{L} = (\boldsymbol{r}\times\boldsymbol{\pi})-(\mu/cr)\boldsymbol{r}$, commute with the Dirac Hamiltonian for the considered system. $|\mathscr{J}|^2$ and $\mathscr{J}_z$ are compatible invariants and we can show that the wavefunctions obtained above are eigenfunctions for these two operators. In quaternionic form (Hautot 1970)

$$S_x = \tfrac{1}{2}i\hbar\sqrt{-1} \qquad S_y = \tfrac{1}{2}j\hbar\sqrt{-1} \qquad S_z = \tfrac{1}{2}k\hbar\sqrt{-1} \tag{15}$$

therefore

$$\mathscr{J}_z = -\hbar\sqrt{-1}\partial_\phi + \tfrac{1}{2}k\hbar\sqrt{-1} \tag{16}$$

and

$$\mathscr{J}_z\psi_{n,\kappa,\jmath,M} = M\hbar\psi_{n,\kappa,\jmath,M}. \tag{17}$$

Moreover, if, following Hautot (1970), we define $\mathscr{L}_3$ by

$$\mathscr{L}_3 = i\mathscr{L}_x + j\mathscr{L}_y + k\mathscr{L}_z$$

it is clear that

$$|\mathscr{J}|^2 = -\mathscr{L}_3\mathscr{L}_3 + 2\hbar\sqrt{-1}\mathscr{L}_3 + \tfrac{3}{4}\hbar^2.$$

Now

$$\frac{\sqrt{-1}}{\hbar}\mathscr{L}_3 = j\,e^{-k\phi}\partial_\theta - i\,e^{-k\phi/2}\,e^{-j\theta}\,e^{-k\phi/2}\frac{1}{\sin\theta}\partial_\phi - i\,e^{-k\phi}\mu\frac{\sqrt{-1}}{c\hbar\sin\theta}.$$

From equations (6) and (12)

$$\mathscr{L}_3\psi_{n,\kappa,\jmath,M} = \hbar\sqrt{-1}\psi_{n,\kappa,\jmath,M}(1+k\kappa\sqrt{-1}) - \frac{\mu}{c}k\,e^{k\phi/2}\,e^{-j\theta}\,e^{-k\phi/2}\psi_{n,\kappa,\jmath,M}$$

and consequently

$$|\mathscr{J}|^2\psi_{n,\kappa,\jmath,M} = \hbar^2\left\{\kappa^2 + \left(\frac{\mu}{c\hbar}\right)^2 - \frac{1}{4}\right\}\psi_{n,\kappa,\jmath,M} = \hbar^2\jmath(\jmath+1)\psi_{n,\kappa,\jmath,M}. \tag{18}$$

The O(3) invariance symmetry generated by $\mathscr{J}$ provides no information about the radial part of the wavefunctions. However, the use of the broken symmetry O(2, 1) (Bacry and Richard 1967, Crubellier and Feneuille 1971) for the radial wave equations (13) is still possible even if μ is not zero, because the fact that κ is an integer for the usual hydrogen atom is not essential in this theory. Thus, all the results obtained by this method and concerning especially radial matrix elements are still valid provided that we replace $(\jmath+\frac{1}{2})$ by $\{(\jmath+\frac{1}{2})^2-(\mu/c\hbar)^2\}^{1/2}$.

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